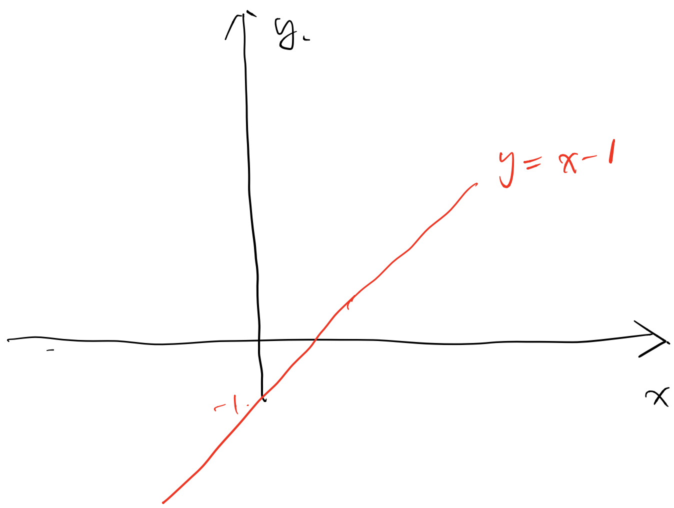
Jan 21

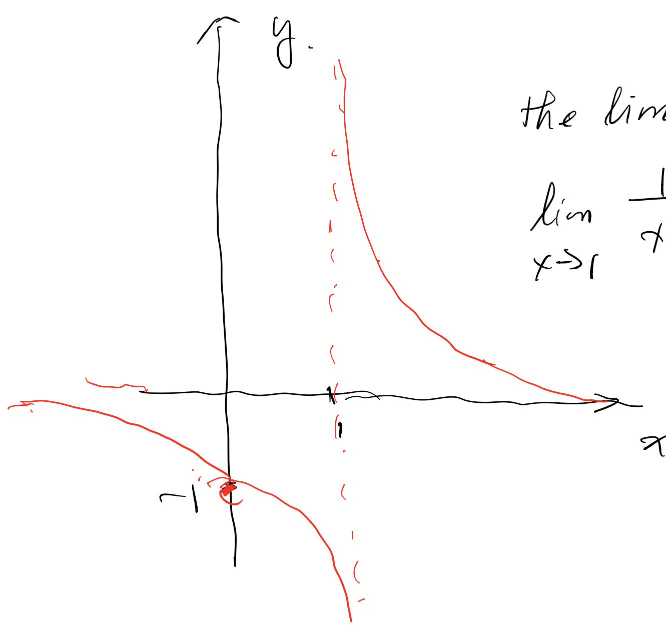
E.g. (Exercise 2.1.1)

$$\lim_{x\to1}\frac{1}{x-1} \overset{?}{\underset{(4)}{=}} \frac{\lim_{x\to1}1}{\lim_{x\to1}(x-1)} = \frac{1}{0}$$

so. rule 4) does not apply here

doesn't make sense.





the limit $\lim_{x\to1}\frac{1}{x-1}$ doesn't exist "DNE"

Example 2.1.7

$$f(x) = \frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{(\sqrt{x})^2-1} = \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)}$$

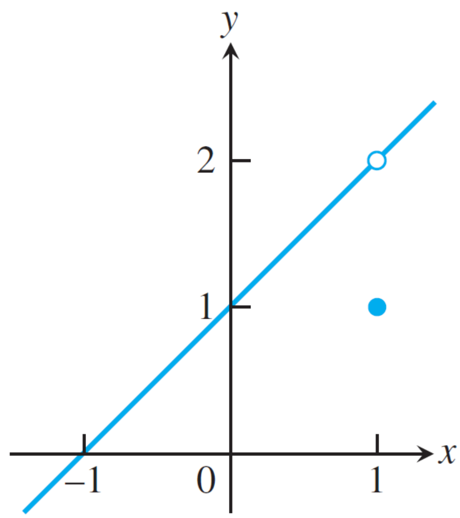
$$= \frac{1}{\sqrt{x}+1} \quad \text{when } x \neq 1$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(\frac{1}{\sqrt{x}+1}\right) \overset{\text{rule (4)}}{=} \frac{\lim_{x\to 1} 1}{\lim_{x\to 1}(\sqrt{x}+1)}$$

$$= \frac{1}{\lim_{x\to 1}\sqrt{x}+1} \overset{\text{rule (5)}}{=} \frac{1}{\sqrt{\lim_{x\to 1} x}+1}$$

$$= \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

D.



Proposition 1.

1. If $f(x) = k$ is a constant function, then
$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k.$$
For instance, $\lim_{x\to 1} 9 = 9$.

2. If $f(x) = x$, then
$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c.$$
For instance, $\lim_{x\to 3} x = 3$.

Proposition 2. **(Algebraic properties of limits, +,−,×, ÷)**

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist (important!), then

1. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
3. $\lim_{x\to c}(f(x)g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

   Especially, $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$ for any constant k

4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$.

   might not $= f(c)$

   might not $= g(c)$

5. $\lim_{x\to c}(f(x))^p = \left[\lim_{x\to c} f(x)\right]^p$ if $\left[\lim_{x\to c} f(x)\right]^p$ exists

Example 2.1.4. Compute the following limits:

1. $\lim_{x\to 1}(x^3+2x-5)$
2. $\lim_{x\to 2}\dfrac{x^4+x^2-1}{x^2+5}$
3. $\lim_{x\to -2}\sqrt{4x^2-3}$

Solution.

1. $\lim_{x\to 1}(x^3+2x-5) = \lim_{x\to 1}x^3 + \lim_{x\to 1}2x - \lim_{x\to 1}5 = 1^3+2\cdot 1-5 = -2.$
2. $\lim_{x\to 2}\dfrac{x^4+x^2-1}{x^2+5} \overset{?}{=} \dfrac{\lim_{x\to 2}(x^4+x^2-1)}{\lim_{x\to 2}(x^2+5)} = \dfrac{\overbrace{\lim_{x\to 2}x^4}^{16} + \overbrace{\lim_{x\to 2}x^2}^{4} - \overbrace{\lim_{x\to 2}1}^{1}}{\lim_{x\to 2}x^2+\lim_{x\to 2}5} = \dfrac{19}{9}.$ ← not zero so can apply (4)

   $4+5$

3. $\lim_{x\to -2}\sqrt{4x^2-3} = \sqrt{\lim_{x\to -2}(4x^2-3)} = \sqrt{\lim_{x\to -2}4x^2 - \lim_{x\to -2}3} = \sqrt{16-3} = \sqrt{13}.$

   $\lim_{x\to 2} f(x)^{\frac{1}{2}}$ $4(4)-3=13$ rule (5) applies ∎

Remark. Generalizing the arguments for the first example above: the limit of any polynomial function $P(x)$,

$$\lim_{x\to c}P(x)=P(c).$$

Exercise 2.1.1. Compute the following limits:

$$\lim_{x\to 1}\frac{1}{x-1};\qquad \lim_{x\to 1}\left(x^2-\frac{3x}{x+5}\right)$$

Example 2.1.5. (Cancelling a common factor)
Find the limit:

$$\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2}.$$

$\overset{?}{=} \dfrac{\lim_{x\to 1}(x^2-1)}{\lim_{x\to 1}(x^2-3x+2)} = \dfrac{0}{0}$ ← doesn't make sense

Solution. We can't directly use property of division of limit because the denominator $\lim_{x\to 1}(x^2-3x+2)=1^2-3\times 1+2=0.$ (4)

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2} &= \lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x-2)}\\ &= \lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-2)}\\ &= \lim_{x\to 1}\frac{x+1}{x-2}\\ = \frac{\lim_{x\to 1}(x+1)}{\lim_{x\to 1}(x-2)} &= \frac{\overbrace{1+1}^{=2}}{\underbrace{1-2}_{=-1\neq 0}} = -2.\end{aligned}$$

← defined when $x\neq 1$ and when $x\neq 1$, the common factor $(x-1)$ can be cancelled.

so rule (4) applies

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Example 2.1.6. Compute

$$\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^2 + 2x - 3}.$$

Solution. Write $p(x) = x^3 - 5x + 4$ and $q(x) = x^2 + 2x - 3$. Because $p(1) = q(1) = 0$, $x - 1$ is a factor of $p(x)$ and $q(x)$. We obtain

$$p(x) = (x-1)(x^2 + x - 4) \text{ and } q(x) = (x-1)(x+3).$$

Then

$$\begin{aligned}
\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^2 + 2x - 3} &= \lim_{x\to 1} \frac{(x-1)(x^2 + x - 4)}{(x-1)(x+3)} \\
&= \lim_{x\to 1} \frac{x^2 + x - 4}{x+3} \\
&= \frac{1^2 + 1 - 4}{1+3} = -\frac{1}{2}.
\end{aligned}$$

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try to apply (4)

Example 2.1.7. (Rationalization)

Let $f : [0,\infty)\backslash\{1\} \to \mathbf{R}$ defined by $f(x) = \frac{\sqrt{x} - 1}{x - 1}$. Find $\lim_{x\to 1} f(x)$.

$\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right) \overset{?}{=} \frac{\lim_{x\to 1}(\sqrt{x}-1)}{\lim_{x\to 1}(x-1)}$

$\underset{0}{\shortparallel}$

$= (\sqrt{x})^2 - 1 = (\sqrt{x}+1)(\sqrt{x}-1)$

so (4) does not apply here

Solution. For $x \neq 1$.

$$\frac{\sqrt{x} - 1}{x - 1} = \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \frac{x - 1}{(x-1)(\sqrt{x} + 1)} = \frac{1}{\sqrt{x} + 1}.$$

Hence

$$\lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x\to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}.$$

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Example 2.1.8. (Rationalization and Cancellation)
Find

$$\lim_{x\to 1} \frac{\sqrt{x} - 1}{x^2 - 1}.$$

Solution.

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{x} - 1}{x^2 - 1} &= \lim_{x\to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x+1)(x-1)(\sqrt{x} + 1)} \\
&= \lim_{x\to 1} \frac{\cancel{x - 1}}{(x+1)\cancel{(x-1)}(\sqrt{x} + 1)} \\
&= \lim_{x\to 1} \frac{1}{(x+1)(\sqrt{x} + 1)} = \frac{1}{4}.
\end{aligned}$$

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